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Lie Algebras of Classical and Stochastic Electrodynamics

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The Lie algebras associated with infinitesimal symmetry transformations of third-order differential equations of interest to classical electrodynamics and stochastic electrodynamics have been obtained. The structure constants for a general case are presented and the Lie algebra for each particular application is easily achieved. By the method used here it is not necessary to know the explicit expressions of the infinitesimal generators in order to determine the structure constants of the Lie algebra.

1. INTRODUCTION

The significance of symmetry transformations in mathematical and theoretical physics is well known (Sudarshan and Mukunda, 1974; Vianna, 1989; Wybourne, 1974). For example: (i) continuous symmetry groups of given field equations are helpful for obtaining similarity solutions (Bluman and Cole, 1974; Steeb, 1985); (ii) from Noether's theorem (Noether, 1918) a close connection is obtained between the conserved quantities of a Lagrangian dynamical system and those transformations of coordinates and time which preserve the action integral; (iii) the transformations which leave the action integral invariant form a Lie group which itself may be a proper subgroup of the group of transformations leaving the equation of motion invariant (Rio Preto and Vianna, 1987). As physical systems are described in general by second-order differential equations, the symmetries of this class of equation have been studied by several authors (Lutzky, 1978; Anderson and Davison, 1974; Leach, 1980; Mariwalla, 1975; Steinberg and Wolf, 1981; Steeb, 1985).

It is also known that there are physical phenomena which are described by third-order ordinary differential equations. These equations,

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which are related to classical electrodynamics (a free radiating particle, the radiating oscillator, for instance) (Abraham, 1905; Lorentz, 1952; Landau and Lifshitz, 1985) and stochastic electrodynamics (Santos, 1974*a,b*), however, have not received corresponding attention. In consequence, one does not yet have a general study of their symmetry transformations. This is our purpose in the present paper. We consider a class of differential equations given by

$$\dddot{x} = f_0(t) + f_1(t)x + f_2(t)\dot{x} + f_3(t)\ddot{x}$$

and we obtain their infinitesimal symmetry transformations and the associated Lie algebras.

Several methods (Rio Preto and Vianna, 1987; Aguirre and Krause, 1984; Lutzky, 1979; Moreira *et al.*, 1985) can be found in the current literature to study invariance properties of differential equations. We shall base our study on the procedure introduced by Aguirre and Krause (1984) in the discussion of the Newtonian one-dimensional linear systems and extended by us (Soares Neto and Vianna, 1988) in the analysis of the Langevin equation. In this context one must observe that while for the second-order ordinary differential equation a Lie group has no more than eight essential parameters (Aguirre and Krause, 1984), for a third-order ordinary differential equation, the maximum number of essential parameters, if the equation admits a continuous symmetry group, is seven (Cohen, 1931).

This paper is organized as follows. In Section 2, we obtain the fundamental equations for the determination of the infinitesimal symmetry transformations of a given third-order ordinary differential equation; in this section we also present the general equations to determine the structure constants of the Lie algebra and to obtain the infinitesimal generators of each particular differential equation with the knowledge of the functions $f_i(t)$ and their derivatives. In Section 3, we apply our equations to some specific problems. For the sake of completeness we consider some purely mathematical examples and, in order to test the method, we analyze an equation which has been studied by other methods (Hill, 1951; Oliveira, 1974), that is, $\dddot{x} = 0$. In fact, this is the only case of the class of equations considered here which has been studied in the literature by other authors and we compare our results with those obtained by them. Finally, Section 4 contains some concluding remarks.

2. THE SYMMETRY EQUATIONS

We are interested in the symmetry transformations of the following class of third-order differential equations:

$$\dddot{x} = f_0(t) + f_1(t)x + f_2(t)\dot{x} + f_3(t)\ddot{x} \tag{1}$$

where $f_i(t)$ $(i = 0, 1, \ldots, 3)$ are differentiable functions and

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, \qquad \dddot{x} = \frac{d^3x}{dt^3}$$

According to the Lie group theory, the symmetry transformations have the general form expressed by

$$t' = t + \varepsilon\eta(t, x) \tag{2}$$

and

$$x' = x + \varepsilon\theta(t, x) \tag{3}$$

Here $0 < \varepsilon \ll 1$. The functions $\eta(t, x)$ and $\theta(t, x)$ are still undefined quantities and will be determined by the invariance condition (1) under the transformations given by (2) and (3). In the following, we will denote the partial derivatives of $\eta(t, x)$ and $\theta(t, x)$ by $\eta_t(t, x) = \partial\eta(t, x)/\partial t$, $\theta_x(t, x) = \partial\theta(t, x)/\partial x$, etc. The transformed equation is

$$\dddot{x}' = f_0(t') + f_1(t')x' + f_2(t')\dot{x}' + f_3(t')\ddot{x}' \tag{4}$$

Inserting equations (2) and (3) into equation (4), expanding the functions $f_i(t')$ $(i = 0, 1, 2, 3)$ up to the first order of the parameter ε, and separating the resulting equation according to the exponents of $\ddot{x}$, we obtain

$$\begin{aligned}
&\eta_{xxx}\dot{x}^4 - 3\eta_{xxt}\dot{x}^3 - 3\eta_{xtt}\dot{x}^2 - 4f_0(t)\eta_x\dot{x} - 4f_1(t)\eta_x x\dot{x} \\
&-3f_2(t)\eta_x\dot{x}^2 - \eta_{ttt}\dot{x} - 3f_0(t)\eta_t - 3f_1(t)\eta_t x - 2f_2(t)\eta_t\dot{x} \\
&+\theta_{xxx}\dot{x}^3 + 3\theta_{xxt}\dot{x}^2 + 3\theta_{xtt}\dot{x} + f_0(t)\theta_x + f_1(t)\theta_x x \\
&+\theta_{ttt} - \dot{f}_0(t)\eta - f_1(t)\theta - \dot{f}_1(t)\eta x - f_2(t)\theta_t - \dot{f}_2(t)\eta\dot{x} \\
&+f_3(t)\eta_{xx}\dot{x}^3 + 2f_3(t)\eta_{xt}\dot{x}^2 + f_3(t)\eta_{tt}\dot{x} - f_3(t)\theta_{xx}\dot{x}^2 \\
&-2f_3(t)\theta_{xt}\dot{x} - f_3(t)\theta_{tt} = 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
&-6\eta_{xx}\dot{x}^2 - 4f_3(t)\eta_x\dot{x} - 3\eta_{tt} - 3f_3(t)\eta_t + 3\theta_{xx}\dot{x} \\
&\quad+3\theta_{xt} + f_3(t)\theta_x - 9\eta_{xt}\dot{x} + 3f_3(t)\eta_x\dot{x} + 2f_3(t)\eta_t + \dot{f}_3(t)\eta - f_3(t)\theta_x = 0
\end{aligned} \tag{6}$$

and

$$3\eta_x = 0 \tag{7}$$

Equation (7) has a straightforward solution

$$\eta(t, x) = \phi_1(t) \tag{8}$$

Inserting equation (8) into equation (6) and collecting the coefficients of $(\dot{x})^0$ and $\dot{x}$, we obtain the following relations:

$$-3\ddot{\phi}_1 - f_3(t)\dot{\phi}_1 + 3\theta_{xt} - \dot{f}_3(t)\phi_1 = 0 \tag{9}$$

and

$$3\theta_{xx} = 0 \tag{10}$$

Solving equation (10), we get

$$\theta(t, x) = \phi_2(t)x + \phi_3(t) \tag{11}$$

Proceeding, first we substitute equation (11) in equation (9) and obtain

$$-3\ddot{\phi}_1 - f_3(t)\dot{\phi}_1 + 3\dot{\phi}_2 - \dot{f}_3(t)\phi_1 = 0 \tag{12}$$

then equations (8) and (11) are inserted into equation (5) and the coefficients of x and $\dot{x}$ give rise to the following equations,

$$\dddot{\phi}_3 - f_2(t)\dot{\phi}_3 - f_1(t)\phi_3 - f_3(t)\ddot{\phi}_3 - 3f_0(t)\dot{\phi}_1 - \dot{f}_0(t)\phi_1 + f_0(t)\phi_2 = 0 \tag{13}$$

$$-\dddot{\phi}_1 + 2f_3(t)\ddot{\phi}_1 - 2f_2(t)\dot{\phi}_1 - \dot{f}_2(t)\phi_1 + 3\ddot{\phi}_2 - 2f_3(t)\dot{\phi}_2 = 0 \tag{14}$$

and

$$\dddot{\phi}_2 - f_3(t)\ddot{\phi}_2 - f_2(t)\dot{\phi}_2 - 3f_1(t)\dot{\phi}_1 - \dot{f}_1(t)\phi_1 = 0 \tag{15}$$

Finally, we derive equation (12) in relation to time, obtaining an expression for $\ddot{\phi}_2(t)$ and substitute it, together with (12), in (15). The result is

$$\dddot{\phi}_1 + \left[\dot{f}_3(t) - \frac{1}{3}f_3^2(t) - f_2(t)\right]\dot{\phi}_1 + \left[\frac{1}{2}\ddot{f}_3(t) - \frac{1}{2}\dot{f}_2(t) - \frac{1}{3}f_3(t)\dot{f}_3(t)\right]\phi_1(t) = 0 \tag{16}$$

Equations (12), (13) and equations (15), (16) are the fundamental equations for the determination of the symmetry transformations. We seek general expressions which will make it possible to obtain the structure constants of the Lie algebra of each particular differential equation with the knowledge of the functions $f_i(t)$ and their derivatives at $t = 0$. In order to do so, we introduce some definitions. The infinitesimal generators of a Lie algebra in an (x, t) realization are defined as

$$X_a(t, x) = \eta_a(t, x)\frac{\partial}{\partial t} + \theta_a(t, x)\frac{\partial}{\partial x} \tag{17}$$

The maximum number of generators of a Lie algebra of a third-order differential equation is seven (Cohen, 1931). In this case, the index a of (17) runs from 1 to 7. The generators obey the following relations:

$$[X_a, X_b] = f^c_{ab} X_c, \qquad a, b, c = 1, 2, \ldots, 7 \tag{18}$$

where $[X_a, X_b]$ represents the commutator of X_a and X_b and the quantities f^c_{ab} are the structure constants of the group.

Using equations (8) and (11), we obtain

$$\eta_a = \phi_{1.a} \tag{19}$$

$$\theta_a = \phi_{2.a} x + \phi_{3.a} \tag{20}$$

Substituting equations (19) and (20) in equation (18) and collecting the coefficients of x^0 and x, we derive the following equations:

$$f^c_{ab} \phi_{1.c} = \phi_{1.a} \dot{\phi}_{1.b} - \phi_{1.b} \dot{\phi}_{1.a} \tag{21}$$

$$f^c_{ab} \phi_{2.c} = \phi_{1.a} \dot{\phi}_{2.b} - \phi_{1.b} \dot{\phi}_{2.a} \tag{22}$$

$$f^c_{ab} \phi_{3.c} = \phi_{1.a} \dot{\phi}_{3.b} + \phi_{3.a} \phi_{2.b} - \phi_{1.b} \dot{\phi}_{3.a} - \phi_{3.b} \phi_{2.a} \tag{23}$$

The first and second derivatives of equation (22) give us the remaining equations,

$$f^c_{ab} \dot{\phi}_{1.c} = \phi_{1.a} \ddot{\phi}_{1.b} - \phi_{1.b} \ddot{\phi}_{1.a} \tag{24}$$

$$f^c_{ab} \ddot{\phi}_{1.c} = \dot{\phi}_{1.a} \ddot{\phi}_{1.b} + \phi_{1.a} \dddot{\phi}_{1.b} - \dot{\phi}_{1.b} \ddot{\phi}_{1.a} - \phi_{1.b} \dddot{\phi}_{1.a} \tag{25}$$

$$\begin{aligned} f^c_{ab} \dot{\phi}_{3.c} = {} & \dot{\phi}_{1.a} \dot{\phi}_{3.b} + \phi_{1.a} \ddot{\phi}_{3.b} + \dot{\phi}_{3.a} \phi_{2.b} + \phi_{3.a} \dot{\phi}_{2.b} \\ & - \dot{\phi}_{1.b} \dot{\phi}_{3.a} - \phi_{1.b} \ddot{\phi}_{3.a} - \dot{\phi}_{3.b} \phi_{2.a} - \phi_{3.b} \dot{\phi}_{2.a} \end{aligned} \tag{26}$$

$$\begin{aligned} f^c_{ab} \ddot{\phi}_{3.c} = {} & \ddot{\phi}_{1.a} \dot{\phi}_{3.b} + 2\dot{\phi}_{1.a} \ddot{\phi}_{3.b} + \phi_{1.a} \dddot{\phi}_{3.b} + \ddot{\phi}_{3.a} \phi_{2.b} \\ & + 2\dot{\phi}_{3.a} \dot{\phi}_{2.b} + \phi_{3.a} \ddot{\phi}_{2.b} - \ddot{\phi}_{1.b} \dot{\phi}_{3.a} - 2\dot{\phi}_{1.b} \ddot{\phi}_{3.a} - \phi_{1.b} \dddot{\phi}_{3.a} \\ & - \ddot{\phi}_{3.b} \phi_{2.a} - 2\dot{\phi}_{3.b} \dot{\phi}_{2.a} - \phi_{2.b} \phi_{2.a} \end{aligned} \tag{27}$$

So far, we have concluded that the general expressions for $\eta(t, x)$ and $\theta(t, x)$ are

$$\begin{aligned} \eta(t, x) = {} & q^1 \phi_{1.1} + q^2 \phi_{1.2} + q^3 \phi_{1.3} + q^4 \phi_{1.4} \\ & + q^5 \phi_{1.5} + q^6 \phi_{1.6} + q^7 \phi_{1.7} \end{aligned} \tag{28}$$

$$\begin{aligned} \theta(t, x) = {} & q^1(\phi_{2.1} x + \phi_{3.1}) + q^2(\phi_{2.2} x + \phi_{3.2}) + q^3(\phi_{2.3} x + \phi_{3.3}) \\ & + q^4(\phi_{2.4} x + \phi_{3.4}) + q^5(\phi_{2.5} x + \phi_{3.5}) + q^6(\phi_{2.6} x + \phi_{3.6}) \\ & + q^7(\phi_{2.7} x + \phi_{3.7}) \end{aligned} \tag{29}$$

Here q^a ($a = 1, 2, \ldots, 7$), are arbitrary constants and we choose them to be

$$\begin{aligned} q^1 &= \eta(0, 0) \\ q^2 &= \eta_t(0, 0) \\ q^3 &= \eta_{tt}(0, 0) \\ q^4 &= \theta(0, 0) \\ q^5 &= \theta_t(0, 0) \\ q^6 &= \frac{1}{2} q_{tt}(0, 0) \\ q^7 &= \theta_x(0, 0) \end{aligned} \tag{30}$$

Table I. The Structure Constants of $\ddot{x} = f_0(t) + f_1(t)x + f_2(t)\dot{x} + f_3(t)\ddot{x}$[a]

$f^1_{12} = 1$					
	$f^2_{13} = 1$				
$f^3_{12} = f_2(0)$					
$+\frac{1}{3} f^2_3(0)$					
$-\dot{f}_3(0)$					
	$f^3_{23} = 1$				
			$f^4_{15} = 1$		
					$f^4_{47} = 1$
				$f^5_{16} = 2$	
		$f^5_{14} = -\dot{f}_3(0)/3$			
		$f^5_{24} = -f_3(0)/3$	$f^5_{25} = 1$		
		$f^5_{34} = -1$			
					$f^5_{57} = 1$
	$f^6_{12} = 3f_0(0)/2$	$f^6_{14} = f_1(0)/2$	$f^6_{15} = f_2(0)/2$	$f^6_{16} = f_3(0)$	$f^6_{17} = -f_0(0)/2$
		$+\ddot{f}_3(0)/12$	$-\dot{f}_3(0)/3$	$f^6_{26} = 2$	
		$-f_3(0)\dot{f}_3(0)/6$	$f^6_{25} = -f_3(0)/3$		
		$-\dot{f}_2(0)/4$	$f^6_{35} = -1/2$		
		$f^6_{24} = \dot{f}_3(0)/6$			
		$-f^2_3(0)/6$			
		$-f_2(0)/2$			
		$f^6_{34} = -f_3(0)/6$			
					$f^6_{67} = 2$
	$f^7_{12} = f_3(0)/3$	$f^7_{13} = 1$			

[a]We denote the derivative of $f_i(t)$ at $t = 0$ by $\dot{f}_i(0)$.

Equations (21)–(27) are identities which hold for all t; hence we can consider them at $t = 0$. In consequence, substituting the initial conditions (30) in (28) and (29) and inserting the resulting expression into (21)–(27) for $t = 0$, we obtain the equations for the structure constants. The equations depend upon $\dddot{\phi}_{1.a}$, $\phi_{1.b}$, $\dot{\phi}_{1.b}$, $\ddot{\phi}_{1.b}$, etc., and we replace them by the values defined by (13), (14) and (16). The final equations are easily solved for the structure constants f^{c}_{ab} and we show them in Table I. However, this table must be used with care! Equation (15) has not been used in the derivation of the structure constants and, as we will see in the examples, it plays the role of a constraint for some choices of the functions $f_i(t)$. The correct way of using Table I is as follows: (a) equation (16) must be solved and the expression for $\phi_1(t)$ obtained, (b) equation (12) is then solved for $\phi_2(t)$ and, finally, (c) $\phi_1(t)$ and $\phi_2(t)$ must be used together with (15) to verify whether there are some null solutions. If not, Table I gives the structure constants straightforwardly. In the positive case, (15) will also point out which generators are null and we should pick up the structure constants from Table I excluding those associated with the null generators. In order to obtain the explicit expression for the generators of the algebra, equation (14) should also be solved.

3. APPLICATIONS

In this section we address a couple of specific examples. In the first one, we go through all steps proposed in the previous section in order to make clear the procedure. In the remaining, we focus on particular points to show some features of either the method or the system being analyzed.

3.1. $\ddot{x} = 0$

Hill (1951) and Oliveira (1974) have already studied this equation using different approaches from the present one. To our knowledge, this is the only equation of the class analyzed here for which the Lie algebra has been obtained previously. In this example $f_i(0) = 0$ for $i = 0, 1, 2, 3$, and, the fundamental equations (13), (14), and (16) become

$$\dot{\phi}_2(t) = \ddot{\phi}_1(t) \tag{31}$$

$$\dddot{\phi}_3(t) = 0 \tag{32}$$

$$\dddot{\phi}_1(t) = 0 \tag{33}$$

The equation for the constraint is

$$\dddot{\phi}_2(t) = 0 \tag{34}$$

We see that the solutions of equations (31)–(33) are also solutions of equation (34). Therefore, for this example, the dimension of the algebra is seven. The expressions for $\eta(t, x)$ and $\theta(t, x)$, by using equations (8) and (11) and the solution of (31)–(34), are

$$\eta(t, x) = at^2 + bt + c \tag{35}$$

$$\theta(t, x) = 2atx + dx + et^2 + ft + g \tag{36}$$

where the arbitrary constants $a, \ldots, g$ are determined from equations (30). Using in equations (35) and (36) the initial conditions defined by (30), we obtain from (17) the generators [see equations (19), (20), (28), and (29)]

$$X_1 = \frac{\partial}{\partial t} \tag{37}$$

$$X_2 = t\frac{\partial}{\partial t} \tag{38}$$

$$X_3 = \frac{1}{2}t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} \tag{39}$$

$$X_4 = \frac{\partial}{\partial x} \tag{40}$$

$$X_5 = t\frac{\partial}{\partial x} \tag{41}$$

$$X_6 = t^2\frac{\partial}{\partial x} \tag{42}$$

$$X_7 = x\frac{\partial}{\partial x} \tag{43}$$

The generators obtained by Hill (1951) are a linear combination of those showed by equations (37)–(43). We show the Lie algebra in Table II.

Table II. Lie Algebra of $\ddot{x} = 0$[a]

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	X_1	X_2+X_7	0	X_4	$2X_5$	0
X_2	$-X_1$	0	X_3	0	X_5	$2X_6$	0
X_3	$-X_2-X_7$	$-X_3$	0	$-X_5$	$-\frac{1}{2}X_6$	0	0
X_4	0	0	X_5	0	0	0	X_4
X_5	$-X_4$	$-X_5$	$\frac{1}{2}X_6$	0	0	0	X_5
X_6	$-2X_5$	$-2X_6$	0	0	0	0	X_6
X_7	0	0	0	$-X_4$	$-X_5$	$-X_4$	0

[a]The generators X_4, X_5, and X_6 form an Abelian subalgebra.

3.2. $\dddot{x} = k\dot{x}$

Here we show an equation for which the first derivative of x has been included. As in the previous case, this example is purely mathematical and there is no physical interpretation for it. The fundamental equations (12), (13), and (16) are

$$\dot{\phi}_2 = \ddot{\phi}_1 \tag{44}$$

$$\dddot{\phi}_3 - k\ddot{\phi}_3 = 0 \tag{45}$$

$$\dddot{\phi}_1 - k\ddot{\phi}_1 = 0 \tag{46}$$

The constraint in this case is

$$\dddot{\phi}_2 - k\ddot{\phi}_2 = 0 \tag{47}$$

The seven solutions of equations (44)–(46) are also solutions of equation (47). Table III shows the Lie algebra associated with this equation. A straightforward calculation determines the functions $\eta(t, x)$ and $\theta(t, x)$ and the infinitesimal generators.

3.3. $\dddot{x} = k\ddot{x}$. A Free Radiating Particle

The inclusion of $\dddot{x}$ gives us the possibility of a physical interpretation for this third-order differential equation. In this case, the particle is free from an external force, but suffers a process of negative acceleration due to the radiation. We have

$$f_3(t) = k \tag{48}$$

$$f_0(t) = f_1(t) = f_2(t) = 0 \tag{49}$$

Table III. Lie Algebra of $\dddot{x} = k\dot{x}$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	$X_1 + kX_3$	$X_2 + X_7$	0	$X_4 + \frac{1}{2}kX_6$	$2X_5$	0
X_2	$-X_1 - kX_3$	0	X_3	$-\frac{1}{2}kX_1$	X_5	$2X_6$	0
X_3	$-X_2 - X_7$	$-X_3$	0	$-X_5$	$-\frac{1}{2}X_6$	0	0
X_4	0	$\frac{1}{2}kX_6$	X_5	0	0	0	X_4
X_5	$-X_4 - \frac{1}{2}kX_6$	$-X_5$	$\frac{1}{2}X_6$	0	0	0	X_5
X_6	$-2X_5$	$-2X_6$	0	0	0	0	X_6
X_7	0	0	0	$-X_4$	$-X_5$	$-X_6$	0

The equations for ϕ_1, ϕ_2, and ϕ_3 are

$$\dddot{\phi}_1 - \frac{1}{3}k^2\dot{\phi}_1 = 0 \tag{50}$$

$$3\dot{\phi}_2 = 3\ddot{\phi}_1 + k\dot{\phi}_1 \tag{51}$$

$$\dddot{\phi}_3 - k\ddot{\phi}_3 = 0 \tag{52}$$

The expression for the constraint is

$$\dddot{\phi}_2 - k\ddot{\phi}_2 = 0 \tag{53}$$

Solving equations (50)–(52), we obtain,

$$\phi_1 = a\exp\left(\frac{k\sqrt{3}t}{3}\right) + b\exp\left(\frac{-k\sqrt{3}t}{3}\right) + c \tag{54}$$

$$\phi_2 = a\frac{(1+\sqrt{3})k}{3}\exp\left(\frac{k\sqrt{3}t}{3}\right) + b\frac{(1-\sqrt{3})k}{3}\exp\left(\frac{-k\sqrt{3}t}{3}\right) + d \tag{55}$$

$$\phi_3 = fe^{kt} + gt + h \tag{56}$$

where $a, \ldots, h$ are arbitrary constants. Inserting equation (55) into equation (53), we obtain

$$a = b = 0 \tag{57}$$

Finally, using the initial conditions (30), we get the expressions for $\eta(x, t)$ and $\theta(x, t)$,

$$\eta(t, x) = q^1 \tag{58}$$

$$\theta(t, x) = q^7 x + \left(\frac{2}{k^2}e^{kt} - \frac{2}{k}t - \frac{2}{k^2}\right)q^6 + q^5 t \tag{59}$$

This is an example where equation (15) has restricted the number of the group parameters. The equation for a free radiating particle has an associated Lie algebra of only five generators. The algebra is shown in Table IV. In this particular example, we notice the presence of an Abelian subalgebra formed by X_1 and X_4.

3.4. $\dddot{x} - k_3\ddot{x} - k_1 x = 0$. The Radiating Oscillator

If we consider

$$f_1(t) = k_1 = -\frac{3}{2}\frac{m^3c^3\omega_0^2}{e^2} \tag{60}$$

and

$$f_3(t) = k_3 = -\frac{3}{2}\frac{m^3c^3}{e^2} \tag{61}$$

Table IV. The Lie Algebra Associated to the Equation with the Free Radiating Particle Equation[a]

	X_1	X_4	X_5	X_6	X_7
X_1	0	0	X_4	$2X_5+kX_6$	0
X_4	0	0	0	0	X_4
X_5	$-X_4$	0	0	0	X_5
X_6	$-2X_5-kX_6$	0	0	0	X_6
X_7	0	$-X_4$	$-X_5$	$-X_6$	0

[a]Equation (15) has restricted the algebra dimension from seven to five.

the equation of this section will represent a particle of charge e and mass m oscillating with angular frequency ω_0 and radiating.

We also have a fifth-dimensional algebra for this case, and the infinitesimal generators are

$$X_1=\frac{\partial}{\partial t} \tag{62}$$

$$X_4=\left[\frac{m_2m_3^2-m_2^2m_3}{c}e^{m_1t}+\frac{m_1^2m_3-m_1m_3^2}{c}e^{m_2t}+\frac{m_1m_2^2-m_1^2m_2}{c}e^{m_3t}\right]\frac{\partial}{\partial x} \tag{63}$$

$$X_5=\left[\frac{m_2^2-m_3^2}{c}e^{m_1t}+\frac{m_3^2-m_1^2}{c}e^{m_2t}+\frac{m_1^2-m_2^2}{c}e^{m_3t}\right]\frac{\partial}{\partial x} \tag{64}$$

$$X_6=\left[\frac{m_3-m_2}{c}e^{m_1t}+\frac{m_1-m_3}{c}e^{m_2t}+\frac{m_2-m_1}{c}e^{m_3t}\right]\frac{\partial}{\partial x} \tag{65}$$

$$X_7=x\frac{\partial}{\partial x} \tag{66}$$

where

$$m_1=A^{1/3}+B^{1/3}+\frac{k_3}{3} \tag{67}$$

$$m_2=\gamma A^{1/3}+\gamma^2B^{1/3}+\frac{k_3}{3} \tag{68}$$

$$m_3=\gamma^2A^{1/3}+\gamma B^{1/3}+\frac{k_3}{3} \tag{69}$$

The quantities A, B, and γ are defined as

$$A=k_1/2+k_3^3/27+(k_1^2+4k_1k_3^3/27+4k_3^6/729-k_3^6/729)^{1/2} \tag{70}$$

$$B=k_1/2+k_3^3/27-(k_1^2+4k_1k_3^3/27+4k_3^6/729-k_3^6/729)^{1/2} \tag{71}$$

$$\gamma=(-1+i\sqrt{3})/2 \tag{72}$$

Table V. The Algebra of $\dddot{x} - k_3\ddot{x} - k_1 x = 0$

	X_1	X_4	X_5	X_6	X_7
X_1	0	$\frac{1}{2}k_1X_6$	X_4	$2X_5+k_3X_6$	0
X_4	$-\frac{1}{2}k_1X_6$	0	0	0	X_4
X_5	$-X_4$	0	0	0	X_5
X_6	$-2X_5-k_3X_6$	0	0	0	X_6
X_7	0	$-X_4$	$-X_5$	$-X_6$	0

We show the Lie algebra associated with this equation in Table V. The Abelian subalgebra $\{X_4, X_5, X_6\}$ is also present in the fifth-dimensional algebras.

3.5. $\dddot{x} - k_3\ddot{x} - k_1 x = -f_0(t)$. The Fundamental Equation of the Stochastic Electrodynamics

The so-called Langevin equation is obtained for k_1 and k_3 defined by equations (60) and (61), respectively, and

$$f_0(t) = -\frac{3}{2}c^3\left(\sum_{\lambda=1}^{2}\int d^3k\, e_x(\mathbf{k}, \lambda)h(\omega_k)\cos[\omega_k t - \mathbf{k}\cdot\mathbf{x} - i\theta(k, z)]\right) \quad (73)$$

The physical quantities entering this formula are defined in Soares Neto and Vianna (1988), Santos (1974*a*,*b*), and Boyer (1975). Due to the expression for $f_0(t)$, the equation for $\phi_3(t)$ is not solvable. The explicit expressions for the generators cannot be obtained in this case. However, the number of linearly independent solutions is provided by (12), (16), and (15) and the structure constants are obtained directly from Table I. We show the algebra in Table VI. We refer the reader to Soares Neto and Vianna (1988), where we have studied in detail various aspects of the Langevin equation.

Table VI. The Lie Algebra of the Fundamental Equation of Stochastic Electrodynamics

	X_1	X_4	X_5	X_6	X_7
X_1	0	$\frac{1}{2}k_1X_6$	X_4	$2X_5+k_3X_6$	$-\frac{1}{2}f_0(0)X_6$
X_4	$-\frac{1}{2}k_1X_6$	0	0	0	X_4
X_5	$-X_4$	0	0	0	X_5
X_6	$-2X_5-k_3X_6$	0	0	0	X_6
X_7	$\frac{1}{2}f_0(0)X_6$	$-X_4$	$-X_5$	$-X_6$	0

4. CONCLUDING REMARKS

We have obtained the Lie algebra of a class of third-order differential equations of interest to classical electrodynamics and stochastic electrodynamics. The method we have used is general and the structure constants of each specific equation are obtained by solving simple differential equations. We believe that the extension of the present method for nonlinear equations is straightforward. This procedure lends itself to an easy computer implementation using the REDUCE language; first attempts in this direction have been made and we are working on a more ambitious program.

A characteristic feature of the method is that the constant structure (Lie algebra) are determined although the functions $\eta(t, x)$ and $\theta(t, x)$ and the generators $X_a(t, x)$ may be unknown (see, for instance, Section 3.5); this certainly is an advantage of this procedure. In fact, our results (see Table I) show that the Lie algebra of a third-order ordinary differential equation of the type expressed in equation (1) depends exclusively on the initial values $f_0(0), f_1(0), f_2(0), f_3(0), \dot{f}_3(0)$, and $\ddot{f}_3(0)$, whereas the determination of the infinitesimal generators $X_a(t, x)$ requires the knowledge of the functions $f_0(t)$, $f_1(t)$, $f_2(t)$, and $f_3(t)$, their first derivatives, and $\ddot{f}_3(t)$ for all t, in order to solve equations (13)–(16). As different sets of initial conditions correspond to different parametrizations of the group, it follows that the resulting commutation relations also depend on the chosen set of initial conditions. However, this change is inherent in the Lie group formalism and corresponds to a change of the basis of the algebra.

Concluding this section, we stress some important aspects from our viewpoint. The equations in which the coefficient of $\ddot{x}$ is different from zero have a fifth-dimensional Lie algebra, while for those cases which $f_3(t) = 0$, the algebra has seven linearly independent generators. There is a three-dimensional Abelian subalgebra formed by X_4, X_5, and X_6 common to all examples studied here. The three two-dimensional subalgebras (X_7, X_4), (X_7, X_5), and (X_7, X_6) are also present in all cases.

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